1. The Axiom of Choice (AC)

Theorem I.12.1 (*ZF*) *The following are equivalent*

1. *The Axiom of Choice (as in Section I.2).*
2. *Every family of non-empty sets has a choice function.*
3. *Every set can be well-ordered.*
4. $\forall xy(x \preccurlyeq y \vee y \preccurlyeq x)$.
5. *Tukey's Lemma.*
6. *The Hausdorff Maximal Principle.*
7. *Zorn's Lemma.*

For (2)

Definition I.12.2 *Let F be a family of non-empty sets. A* choice function *for F is a function g with* $\mathrm{dom}(g) = F$ *such that $g(x) \in x$ for all $x \in F$. A* choice set *for F is a set C such that $C \cap x$ is a singleton set for all $x \in F$.*

Choice Set Vs Choice Function.

$F = \{\{2\}, \{2,3\}, \{3\}\}$ → No choice set for F.
But $\exists$ choice function. $g(x) = \min(x)$.

(1) ⇔ (2):

(2) → (1) F disjoint. g is a choice function for F
$\Rightarrow C = \{g(x) : x \in F\}$ is a choice set for F.

(1) → (2) $\forall F$ let $F^* = \{\{x\} \times x : x \in F\}$.
$\{x\} \times x = \{(x, i) : i \in x\}$
If $x \neq y$, $\{x\} \times x$ & $\{y\} \times y$ are disjoint.
$\xRightarrow{(1)}$ $\exists$ C choice set for F^*
C ← sets for ordered pairs, a function.
$\Rightarrow$ C is a choice fct for F.

For (5)

Definition I.12.3 *If $\mathcal{F} \subseteq \mathcal{P}(A)$, then $X \in \mathcal{F}$ is* maximal *in $\mathcal{F}$ iff it is maximal with respect to the relation $\subsetneqq$ (see Definition I.7.18); that is, X is not a proper subset of any set in $\mathcal{F}$.*

Definition I.12.5 *$\mathcal{F} \subseteq \mathcal{P}(A)$ is of* finite character *iff for all $X \subseteq A$: $X \in \mathcal{F}$ iff every finite subset of X is in $\mathcal{F}$.*

Definition I.12.7 Tukey's Lemma *is the assertion that whenever $\mathcal{F} \subseteq \mathcal{P}(A)$ is of* finite character *and $X \in \mathcal{F}$, there is a maximal $Y \in \mathcal{F}$ such that $X \subseteq Y$.*

For (6) & (7)

Definition I.12.8 *Let $<$ be a strict partial order of a set A. Then $C \subseteq A$ is a* chain *iff C is totally ordered by $<$; C is a* maximal chain *iff in addition, there are no chains $X \supsetneq C$.*

Definition I.12.9 The Hausdorff Maximal Principle *asserts that whenever $<$ is a strict partial order of a set A, there is a maximal chain $C \subseteq A$.*

Definition I.12.10 Zorn's Lemma *is the assertion that whenever $<$ is a strict partial order of a set A satisfying*

(✌) *For all chains $C \subseteq A$ there is some $b \in A$ such that $x \leq b$ for all $x \in C$,*

then for all $a \in A$, there is a maximal (see Definition I.7.18) $b \in A$ with $b \geq a$.

Assuming AC

$\Downarrow$

2. Cardinal Arithmetic

$\forall\, x \xrightarrow{\text{well-ordered}} |x|$ defined

Definition I.13.1 *If κ, λ are cardinals, then*

☞ $\boxed{\kappa + \lambda} = |\{0\} \times \kappa \cup \{1\} \times \lambda|$

☞ $\boxed{\kappa \cdot \lambda} = |\kappa \times \lambda|$

☞ $\boxed{\kappa^\lambda} = |{}^\lambda\kappa|$

☆ Sometimes, boxes are omitted.

Note

k^λ —
- Ordinal exponent — k^λ
- Cardinal exponent — $\boxed{k^\lambda}$
- Sets of function. — ${}^\lambda k$

Lemma I.13.2 *If $\kappa, \lambda, \kappa', \lambda'$ are cardinals and $\kappa \le \kappa'$ and $\lambda \le \lambda'$, then $\kappa + \lambda \le \kappa' + \lambda'$, $\kappa \cdot \lambda \le \kappa' \cdot \lambda'$, and $\kappa^\lambda \le (\kappa')^{\lambda'}$ (unless $\kappa = \kappa' = \lambda = 0$), where* cardinal *arithmetic is meant throughout.*

Proof ① $\{0\} \times k \cup \{1\} \times \lambda \subseteq \{0\} \times k' \cup \{1\} \times \lambda'$

② $|k \times \lambda| \subseteq |k' \times \lambda'|$

③ when $k' > 0$

$\varphi : {}^{\lambda}k \xrightarrow{1-1} {}^{(\lambda')}(k')$

with

$\left[\begin{array}{l} \varphi(f) \upharpoonright \lambda = f \\ (\varphi(f))(\xi) = 0. \qquad \lambda \le \xi \le \lambda' \end{array} \right.$

when $k = k' = 0$.

using the idea

$\begin{cases} 0^0 = |{}^{0}0| = |\{\phi\}| = 1. & \phi \to \phi \\ 0^\lambda = |{}^{\lambda}0| = |\phi| = 0 & \text{for } \lambda > 0 \end{cases}$

Laws

Lemma I.13.3 *If κ, λ, θ are cardinals, then using cardinal arithmetic throughout:*

1. $\kappa + \lambda = \lambda + \kappa$.
2. $\kappa \cdot \lambda = \lambda \cdot \kappa$.
3. $(\kappa + \lambda) \cdot \theta = \kappa \cdot \theta + \lambda \cdot \theta$.
4. $\kappa^{(\lambda \cdot \theta)} = (\kappa^\lambda)^\theta$.
5. $\kappa^{(\lambda + \theta)} = \kappa^\lambda \cdot \kappa^\theta$.

Involve both cardinal & ordinal operation.

Lemma I.13.4 *For any ordinals α, β: $|\alpha + \beta| = \boxed{|\alpha| + |\beta|}$ and $|\alpha \cdot \beta| = \boxed{|\alpha| \cdot |\beta|}$.*

e.g. $\alpha = \beta = \omega = |\omega|$: $\quad \underbrace{\omega,\ \omega+\omega,\ \omega\cdot\omega}_{\text{same cardinality}}$

Lemma I.13.5 *If κ, λ are finite cardinals, then $\boxed{\kappa + \lambda} = \kappa + \lambda$, $\boxed{\kappa \cdot \lambda} = \kappa \cdot \lambda$, and $\boxed{\kappa^\lambda} = \kappa^\lambda$.* (since $\boxed{k^{\lambda+1}} = \boxed{k^\lambda} \cdot k \quad = |k+\lambda|$

Idea: finite ordinals are cardinals

Lemma I.13.6 *If κ, λ are cardinals and at least one of them is infinite, then $\boxed{\kappa + \lambda} = \max(\kappa, \lambda)$. Also, if neither of them are 0 then $\boxed{\kappa \cdot \lambda} = \max(\kappa, \lambda)$.*

Now,

- Cardinal { sum, product — finite: reduced to ordinal { sum, product. — infinite: max function
- Cardinal Exponentiation $\rightarrow$ CH

Lemma I.13.7 $2^\kappa = |\mathcal{P}(\kappa)|$ *for every cardinal κ, and $2^{\aleph_\alpha} \geq \aleph_{\alpha+1}$ for every ordinal α. All exponentiation here is cardinal exponentiation.*

Definition I.13.8 *The* Continuum Hypothesis, *CH, is the statement $2^{\aleph_0} = \aleph_1$. The* Generalized Continuum Hypothesis, *GCH, is the statement $\forall\alpha[2^{\aleph_\alpha} = \aleph_{\alpha+1}]$. All exponentiation here is cardinal exponentiation.*

↳ Knowing 2^λ for infinite λ

↳ k^λ. for infinite λ ?

Lemma I.13.9 *If $2 \leq \kappa \leq 2^\lambda$ and λ is infinite, then $\kappa^\lambda = 2^\lambda$. All exponentiation here is cardinal exponentiation.*

↳ Cofinality.

Definition I.13.10 *If γ is any limit ordinal, then the* cofinality *of γ is*

$$\mathrm{cf}(\gamma) = \min\{\mathrm{type}(X) : X \subseteq \gamma \wedge \underline{\sup(X) = \gamma}\} \quad .$$

$\Leftrightarrow$ X is unbounded in γ

γ *is* regular *iff* $\mathrm{cf}(\gamma) = \gamma$.

Eg) $\gamma = \omega^2$

$\omega \cup \{\omega \cdot n : n \in \omega\}$. $\rightarrow$ order type $\omega \cdot 2$, unbounded in γ

$\{\omega \cdot n : n \in \omega\}$ $\rightarrow$ order type ω, unbounded in γ

$\Rightarrow \mathrm{cf}(\gamma) = \omega$. (if $X \subseteq \gamma$ is finite, cannot be unbounded).

Lemma I.13.11 *For any limit ordinal γ:*

1. *If $A \subseteq \gamma$ and $\sup(A) = \gamma$ then $\mathrm{cf}(\gamma) = \mathrm{cf}(\mathrm{type}(A))$.*
2. *$\mathrm{cf}(\mathrm{cf}(\gamma)) = \mathrm{cf}(\gamma)$, so $\mathrm{cf}(\gamma)$ is regular.*
3. *$\omega \leq \mathrm{cf}(\gamma) \leq |\gamma| \leq \gamma$*
4. *If γ is regular then γ is a cardinal.*
5. *If $\gamma = \aleph_\alpha$ where α is either 0 or a successor, then γ is regular.*
6. *If $\gamma = \aleph_\alpha$ where α is a limit ordinal, then $\mathrm{cf}(\gamma) = \mathrm{cf}(\alpha)$.*

Theorem I.12.11 (AC) *Let κ be an infinite cardinal. If $\mathcal{F}$ is a family of sets with $|\mathcal{F}| \leq \kappa$ and $|X| \leq \kappa$ for all $X \in \mathcal{F}$, then $|\bigcup \mathcal{F}| \leq \kappa$.*

Theorem I.13.12 *Let θ be any cardinal.*

1. *If θ is regular and $\mathcal{F}$ is a family of sets with $|\mathcal{F}| < \theta$ and $|S| < \theta$ for all $S \in \mathcal{F}$, then $|\bigcup \mathcal{F}| < \theta$.*
2. *If $\mathrm{cf}(\theta) = \lambda < \theta$, then there is a family $\mathcal{F}$ of subsets of θ with $|\mathcal{F}| = \lambda$ and $\bigcup \mathcal{F} = \theta$, such that $|S| < \theta$ for all $S \in \mathcal{F}$.*

Idea : (1) $X = \{|S| : S \in \mathcal{F}\}$ $\rightarrow$ type$(X) < \theta$ $\rightarrow$ sup$(X) < \theta$

Let $k = \max\{\sup(X), |\mathcal{F}|\} < \theta$.

k finite. ✓

k infinite $\rightarrow$ use Theorem 1.12.11

(2) $\mathcal{F} \subseteq \theta$ and $\mathcal{F}$ of order type λ. s.t. $\sup \mathcal{F} = \bigcup \mathcal{F} = \theta$.

Theorem I.13.13 (König, 1905) *If $\kappa \geq 2$ and λ is infinite, then* $\mathrm{cf}(\kappa^\lambda) > \lambda$.

⇨ Final Form :

Theorem I.13.14 *Assume GCH, and let κ, λ be cardinals with* $\max(\kappa, \lambda)$ *infinite.*

1. *If $2 \leq \kappa \leq \lambda^+$, then $\kappa^\lambda = \lambda^+$.*
2. *If $1 \leq \lambda \leq \kappa$, then κ^λ is κ if $\lambda < \mathrm{cf}(\kappa)$ and κ^+ if $\lambda \geq \mathrm{cf}(\kappa)$.*

3. Axiom of Foundation (AF)

$$\exists y(y \in x) \quad \rightarrow \quad \exists y(y \in x \land \neg \exists z(z \in x \land z \in y))$$

- $\in$ is well-founded on V

$\Leftrightarrow$ $\forall$ non-empty subset x of V, has an $\in$-minimal element y

$\Leftrightarrow$ $\forall x[x \neq \emptyset \rightarrow \exists y \in x(y \cap x = \emptyset)]$.

- Avoid "patheological sets"

Counter example :

(1) $a \in a$, $x = \{a\}$ $\qquad x \cap a = x \neq \phi$

(2). "$\in$" has cycles. $\qquad$ if $a_1 \in a_2 \in \cdots \in a_n \in a_1$, $x = \{a_1, a_2, \cdots a_n\}$,

Theorem I.14.10 .

- $\boxed{V = WF} \Leftrightarrow AF$

↳ the class of well-founded sets

↳ can be obtained from nothing, ϕ.

↳ where all mathematics takes place.

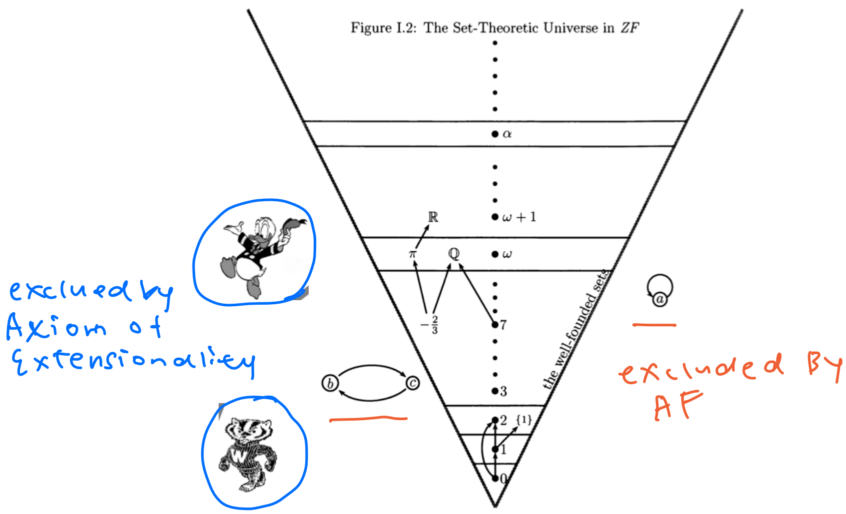

Figure I.2: The Set-Theoretic Universe in ZF

Definition I.14.1 *By recursion on $\alpha \in ON$, define $R(\alpha)$ by:*

1. $R(0) = \emptyset$.
2. $R(\alpha + 1) = \mathcal{P}(R(\alpha))$.
3. $R(\gamma) = \bigcup_{\alpha<\gamma} R(\alpha)$ *for limit ordinals* γ.

Then:

4. $WF = \bigcup_{\delta \in ON} R(\delta) =$ *the class of all well-founded sets.*
5. *The set x is* well-founded *iff* $\exists\delta[x \in R(\delta)]$.
6. *For $x \in WF$:* rank(x) *is the least α such that* $x \in R(\alpha + 1)$.

Table I.2: The First Sixteen Sets

rank	sets
0	$\emptyset = 0$
1	$\{\emptyset\} = 1$
2	$\{\{\emptyset\}\} = \{1\}$, $\{\emptyset, \{\emptyset\}\} = 2$
3	$\{\{1\}\}$, $\{0, \{1\}\}$, $\{1, \{1\}\}$, $\{0, 1, \{1\}\}$, $\{2\}$, $\{0, 2\}$, $\{1, 2\}$, $\{0, 1, 2\} = 3$, $\{\{1\}, 2\}$, $\{0, \{1\}, 2\}$, $\{1, \{1\}, 2\}$, $\{0, 1, \{1\}, 2\}$

$|R(0)| = 0$. $|R(n+1)| = 2^{|R(n)|}$

$|R(1)| = 2^0 = 1$. $|R(2)| = 2^1 = 2$. $|R(3)| = 2^2 = 4$.

$|R(4)| = 2^4 = 16$. $|R(5)| = 2^{16} = 65536$.

Lemma I.14.4

1. *Every $R(\beta)$ is a transitive set.*
2. $\alpha \leq \beta \to R(\alpha) \subseteq R(\beta)$.
3. $R(\alpha + 1) \setminus R(\alpha) = \{x \in WF : \text{rank}(x) = \alpha\}$.
4. $R(\alpha) = \{x \in WF : \text{rank}(x) < \alpha\}$.
5. *If $x \in y$ and y in WF, then $x \in WF$ and* $\text{rank}(x) < \text{rank}(y)$.
   (transitive)

rank of an ordinal

Lemma I.14.5

1. $ON \cap R(\alpha) = \alpha$ *for each* $\alpha \in ON$.
2. $ON \subseteq WF$
3. $\text{rank}(\alpha) = \alpha$ *for each* $\alpha \in ON$.

Eg

ordinal	0	1	2	3
rank	0	1	2	3

rank of a set

Lemma I.14.6 *For any set y: $y \in WF \leftrightarrow y \subseteq WF$, in which case:*

$$\text{rank}(y) = \sup\{\text{rank}(x) + 1 : x \in y\}$$

eg

$\text{rank}(\{2,5\}) = \max\{3,6\} = 6.$

$\text{rank}(\langle 2,5\rangle) = \text{rank}(\{\{2\},\{2,5\}\})$

$= \max\{4,7\} = 7.$

i.e.

Lemma I.14.7 *If $z \subseteq y \in WF$ then $z \in WF$ and* $\text{rank}(z) \leq \text{rank}(y)$.

Lemma I.14.8 *Suppose that $x, y \in WF$. Then:*

1. $\{x, y\} \in WF$ *and* $\text{rank}(\{x, y\}) = \max(\text{rank}(x), \text{rank}(y)) + 1$.
2. $\langle x, y\rangle \in WF$ *and* $\text{rank}(\langle x, y\rangle) = \max(\text{rank}(x), \text{rank}(y)) + 2$.
3. $\mathcal{P}(x) \in WF$ *and* $\text{rank}(\mathcal{P}(x)) = \text{rank}(x) + 1$.
4. $\bigcup x \in WF$ *and* $\text{rank}(\bigcup x) \leq \text{rank}(x)$.
5. $x \cup y \in WF$ *and* $\text{rank}(x \cup y) = \max(\text{rank}(x), \text{rank}(y))$.
6. $\text{trcl}(x) \in WF$ *and* $\text{rank}(\text{trcl}(x)) = \text{rank}(x)$ (*see Definition I.9.5*).

Definition I.14.13 *HF* $= R(\omega)$ *is called the set of* hereditarily finite sets.

countable

all finite mathematics lives

eg. finite ordinals.
$^m n$. $n, m \in \omega$

4. Real Numbers & Symbolic Entities

ZF

$\boxed{\text{Getting } \mathbb{Q}}$

Definition I.15.1 $\mathbb{Q}$ *is the union of* ω *with the set of all* $\langle i, \langle m, n\rangle\rangle \in \omega \times (\omega \times \omega)$ *such that:*

sign digit.
0 +
1 −

1. $m, n \geq 1$
2. $i \in \{0, 1\}$
3. $\gcd(m, n) = 1$ avoid multiple representation
4. *If* $i = 0$ *then* $n \geq 2$ (exclude $7 = <0, <7, 2>>$)

$\in \omega$

$\mathbb{Z} = \omega \cup \{\langle 1, \langle m, 1\rangle\rangle : 0 < m < \omega\}$.

eg.
$$\frac{2}{3} = <0, <2, 3>>$$
$$-\frac{2}{3} = <1, <2, 3>>$$
$$-7 = <1, <7, 1>>$$

$\mathbb{Z} \subseteq \mathbb{Q} \subseteq HF$.
rank $(\mathbb{Z})$ = rank $(\mathbb{Q}) = \omega$.
rank $(-2/3) = 7$.

Definition I.15.3 $+, \cdot,$ *and* $<$ *are defined on* $\mathbb{Q}$ *in the "obvious way", to make* $\mathbb{Q}$ *into an ordered field containing* ω.

(More Axioms of orders are needed).

Getting $\mathbb{R}$ and $\mathbb{C}$

Definition I.15.4 *$\mathbb{R}$ is the set of all $x \in \mathcal{P}(\mathbb{Q})$ such that $x \neq \emptyset$, $x \neq \mathbb{Q}$, x has no largest element, and*

$$\forall p, q \in \mathbb{Q}[p < q \in x \to p \in x] \quad . \qquad (*)$$

$\mathbb{C} = \mathbb{R} \times \mathbb{R}$.

- $x = C_x = \{q \in \mathbb{Q} : q < x\}$. (a subset of $\mathbb{Q}$ satisfying $(*)$)
  if $x \in \mathbb{R}$
- $\mathbb{R}$ = the collection of all sets satisfying $(*)$.
- complex number = a pair of reals $\langle x, y \rangle$.
  $= x + iy$.
- $\mathrm{rank}(x) = \omega$, $\forall x \in \mathbb{R}$.
  $\mathrm{rank}(\mathbb{R}) = \omega + 1$.
  $\mathrm{rank}(\mathbb{C}) = \omega + 3$.

Definition I.15.6 *An* ordered field F *is* Dedekind-complete *iff it satisfies the least upper bound axiom — that is, whenever $X \subseteq F$ is non-empty and bounded above, the least upper bound,* $\sup X$, *exists.*

Proposition I.15.7 *All Dedekind-complete ordered fields are isomorphic.*

These 2 guarantees different constructions of real numbers are the same. (Through. isomorphism).

Getting Symbolic Expressions

Consider a boolean expression

$\sigma = \neg[p \wedge q]$ (a sequence of 6 symbols)

↳ a function domain $6 = \{0, 1, 2, 3, 4, 5\}$
$\sigma(0) = $ "$\neg$"

↳ want to represent symbol by natural numbers

Definition I.15.13 *P_n is the number $2n+2$. The symbols $], [, \neg, \vee, \wedge$ are shorthand for the numbers $1, 3, 5, 7, 9$, respectively.*

$P_0, P_1, \cdots$ → proposition letters / boolean variables

$A = \{1, 3, 5, 7, 9\} \cup \{2n+2 : n \in \omega\} \subseteq \omega$

→ alphabets

no symbols is also a string of symbols.

Definition I.15.16 *Assume that $A \cap A^{<\omega} = \emptyset$, and fix $\tau_0, \dots, \tau_{m-1} \in A \cup A^{<\omega}$. Let σ_i be τ_i if $\tau_i \in A^{<\omega}$, and the sequence of length 1, (τ_i), if $\tau_i \in A$. Then $\tau_0, \dots, \tau_{m-1}$ denotes the string $\sigma_0 ^\frown \cdots ^\frown \sigma_{m-1} \in A^{<\omega}$.*

→ starts the discussion of formal logic.